Dynamic Partial Correlation Models

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Background

- We propose to parameterize the correlation matrix using a sequence of partial correlations which are built recursively from previous partial correlations and pairwise correlations using the so called *D*-vine copula structure.
- A great advantage of this parameterization is that we only have to consider bivariate relationships since partial correlation processes can be specified separately, providing a new family of very flexible multivariate dynamic models.
- The method is easily scalable to higher dimensions without loosing computational stability, and allows for a much simplified asymptotic analysis of the process and the maximum likelihood estimator.

Approach to modeling correlation matrices

- Consider a real-valued N-dimensional time series $\{\boldsymbol{y}_t\}_{t\in\mathbb{Z}}$ and a sequence of corresponding information sets $\mathcal{F}_{t-1} = \{\boldsymbol{y}_{t-1}, \boldsymbol{y}_{t-2}, \dots\}$ with

$$\boldsymbol{y}_t \sim t_\nu\left(\boldsymbol{0}_N, \frac{\nu-2}{\nu}\boldsymbol{R}_t, \nu\right), \qquad (1)$$

- $t_\nu(\boldsymbol{0}_N, \boldsymbol{R}_t, \nu)$ denotes an N-dimensional Student's t distribution with zero conditional mean, conditional covariance matrix $\boldsymbol{R}_t$, and $\nu > 2$ degrees of freedom.
- The link between the pairwise and partial correlations is obtained from Anderson (1958) and Joe (2006) via the recursive formula

$$\rho_{i,j|L_{ij};t} = \frac{\boldsymbol{D}_{0,i,j;t}}{\sqrt{\boldsymbol{D}_{1,i,j,|L_{ij};t}\cdot\boldsymbol{D}_{2,i,j,|L_{ij};t}}},$$

for $i = 1,\dots,N-2$, $j = i+2,\dots,N$, and $L_{ij} = \{i+1,\dots,j-1\}$, where

$$\begin{aligned}\boldsymbol{D}_{0,i,j;t} &= \rho_{i,j;t} - \boldsymbol{r}'_{1,i,j;t}\boldsymbol{R}^{-1}_{i,j;t}\boldsymbol{r}_{3,i,j;t},\\ \boldsymbol{D}_{1,i,j;t} &= \rho_{i,j;t} - \boldsymbol{r}'_{1,i,j;t}\boldsymbol{R}^{-1}_{i,j;t}\boldsymbol{r}_{1,i,j;t},\\ \boldsymbol{D}_{2,i,j;t} &= \rho_{i,j;t} - \boldsymbol{r}'_{3,i,j;t}\boldsymbol{R}^{-1}_{i,j;t}\boldsymbol{r}_{3,i,j;t},\end{aligned} \qquad \text{corr}(\boldsymbol{y}_{i,j;t}) = \begin{bmatrix} 1 & \boldsymbol{r}'_{1,i,j;t} & \rho_{i,j;t}\\ \boldsymbol{r}_{1,i,j;t} & \boldsymbol{R}_{i,j;t} & \boldsymbol{r}_{3,i,j;t}\\ \rho_{i,j;t} & \boldsymbol{r}'_{3,i,j;t} & 1\end{bmatrix},$$

and $\boldsymbol{y}_{i,j;t} = (\boldsymbol{y}_{i,t},\dots,\boldsymbol{y}_{j,t})'$.

- This link can easily be inverted as

$$\rho_{i,j;t} = \boldsymbol{r}'_{1,i,j;t}\boldsymbol{R}^{-1}_{i,j;t}\boldsymbol{r}_{3,i,j;t} + \rho_{i,j|L_{ij};t}\sqrt{\boldsymbol{D}_{1,i,j,|L_{ij};t}\cdot\boldsymbol{D}_{2,i,j,|L_{ij};t}}$$

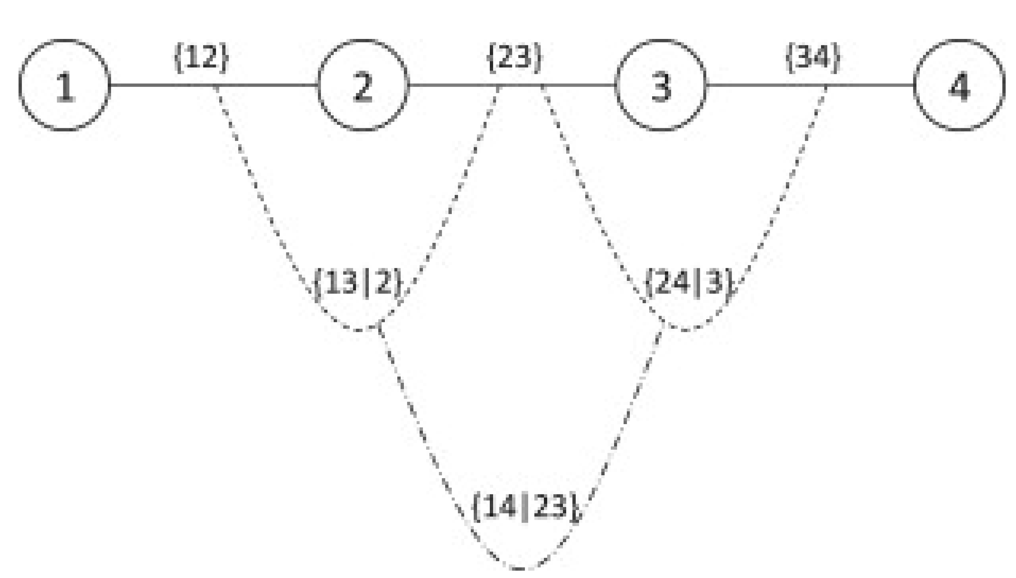

Fig. 1: *D*-vine copula structure.

Dynamics of the partial correlations

- The key step is obtained by the conditional distribution of the bivariate vector $\boldsymbol{y}_{i,j;t} = (y_{i,t}, y_{j,t})'$ in (1) conditional on $\mathcal{F}_{t-1}$ and $\boldsymbol{y}_{L,t} = \{\boldsymbol{y}_{k,t}\}_{k\in L_{ij}}$ is

$$\boldsymbol{y}_{i,j;t} = \begin{pmatrix} y_{i,t}\\ y_{j,t}\end{pmatrix} |\mathcal{F}_{t-1}, \boldsymbol{y}_{L_{ij},t} \sim t_\nu\left(\boldsymbol{\mu}_{i,j|L_{ij};t}, \frac{\nu-2}{\nu}\boldsymbol{R}_{i,j|L_{ij};t}, \nu_{i,j|L_{ij}}\right),$$

(Ding, 2016) with $\nu_{i,j|L_{ij}} = \nu + \#L_{ij} = \nu + j + i - 1$ and

$$\boldsymbol{\mu}_{i,j|L_{ij};t} = \begin{pmatrix}\boldsymbol{r}'_{1,i,j;t}\\ \boldsymbol{r}'_{3,i,j;t}\end{pmatrix}\boldsymbol{R}^{-1}_{i,j;t}\boldsymbol{y}_{i+1:j-1;t} \qquad \boldsymbol{R}_{i,j|L_{ij};t} = \begin{bmatrix} 1 & \rho_{i,j|L_{ij};t}\\ \rho_{i,j|L_{ij};t} & 1\end{bmatrix}.$$

- For $i = 1,\dots,N-2$, $j = i+2,\dots,N$, and $L_{ij} = \{i+1,\dots,j-1\}$, We consider the parametrization $\rho_{i,j|L_{ij};t} = g(f_{i,j|L_{ij};t}) = \tanh(f_{i,j|L_{ij};t})$ for $f_{i,j|L_{ij},t}\in\mathbb{R}$.
- The score expressions are given by

$$\nabla_{i,j|L_{ij};t} = \frac{1}{2}\boldsymbol{G}'_{i,j|L_{ij};t}\boldsymbol{\mathcal{D}}'_2(\boldsymbol{R}^{-1}_{i,j|L_{ij};t}\otimes\boldsymbol{R}^{-1}_{i,j|L_{ij};t}) \times \text{vec}\Big(w_{i,j|L_{ij};t}(\boldsymbol{y}_{i,j;t}-\boldsymbol{\mu}_{i,j|L_{ij};t})(\boldsymbol{y}_{i,j;t}-\boldsymbol{\mu}_{i,j|L_{ij};t})' - \boldsymbol{R}_{i,j|L_{ij};t}\Big),$$

with $\boldsymbol{\mathcal{D}}_2$ as the (4×3) duplication matrix (Magnus and Neudecker, 2019) and

$$w_{i,j|L_{ij};t} = \frac{\nu_{i,j|L_{ij}}+2}{\nu_{i,j|L_{ij}} - 2 + (\boldsymbol{y}_{i,j;t}-\boldsymbol{\mu}_{i,j|L_{ij};t})'\boldsymbol{R}^{-1}_{i,j|L_{ij};t}(\boldsymbol{y}_{i,j;t}-\boldsymbol{\mu}_{i,j|L_{ij};t})}$$

$$\boldsymbol{G}_{i,j|L_{ij};t} = \partial\text{vech}(\boldsymbol{R}^{-1}_{i,j|L_{ij};t})/\partial f_{i,j|L_{ij};t} = \begin{pmatrix}0 & g'(f_{i,j|L_{ij};t}) & 0\end{pmatrix}'$$

- The corresponding Fisher information

$$\mathcal{I}_{i,j|L_{ij};t} = \frac{1}{\nu_{i,j|L_{ij}}+4}\Big((\nu_{i,j|L_{ij}}+2)(1+\rho_{i,j|L_{ij};t}) - 2\rho_{i,j|L_{ij};t}\Big)$$

- The transition dynamics for $f_{i,j|L_{ij};t}$ to $f_{i,j|L_{ij};t+1}$ are given by

$$f_{i,j|L_{ij};t+1} = \omega_{i,j|L_{ij}} + \beta_{i,j|L_{ij}}f_{i,j|L_{ij};t} + \alpha_{i,j|L_{ij}}s_{i,j|L_{ij};t},$$

with $s_{i,j|L_{ij};t} = \mathcal{I}^{-1}_{i,j|L_{ij};t}\nabla_{i,j|L_{ij};t}$.

Maximum likelihood estimation

- The $\log$-likelihood function with $\boldsymbol{\theta} = (\{\omega_{i,j|L_{ij}}, \beta_{i,j|L_{ij}}, \alpha_{i,j|L_{ij}}\}, \nu)'$ is

$$L_T(\boldsymbol{\theta}) = \sum_{t=1}^{T}\ell_t(\boldsymbol{\theta}) := \sum_{t=1}^{T}\Bigg\{\ln\Gamma\left(\frac{\nu+N}{2}\right) - \ln\Gamma\left(\frac{\nu}{2}\right) - \frac{N}{2}\ln((\nu-2)\pi) - \frac{1}{2}\left[\ln|\boldsymbol{R}_t(\boldsymbol{\theta})| + (\nu+N)\ln\left(1+\frac{\boldsymbol{y}'_t\boldsymbol{R}_t^{-1}(\boldsymbol{\theta})\boldsymbol{y}_t}{\nu-2}\right)\right]\Bigg\}.$$

- We prove consistency and asymptotic normality of the MLE $\hat{\boldsymbol{\theta}}$:

$$\sqrt{T}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0) \Rightarrow \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{I}}(\boldsymbol{\theta}_0))$$

Empirical application

- We consider daily data of the market factor *Mkt-RF* proxied by the excess returns on the S&P500 index as well as the FamaFrench size and value factors *SMB* (Small Minus Big) and *HML* (High Minus Low).

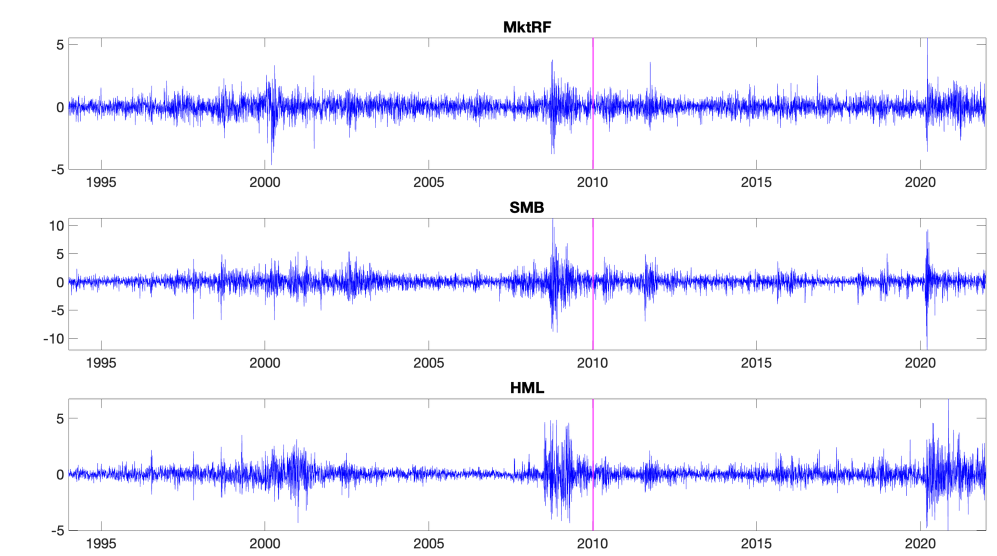

Fig. 2: Daily returns from 3 January 1994 to 31 December 2021.

- We compare our dynamic correlation model against the (classic) DCC model introduced by Engle (2002) and by Aielli (2013), the GAS correlation model of Creal et al. (2011) and the $\log$-Score model of Hafner and Wang (2021).

	t-Partial-Corr	*t-Log-Corr*	*t-GAS*	*t-cDCC*
N.Param	19	19	19	19
$L_T(\hat{\boldsymbol{\theta}}_T)$	$\mathbf{-15604.38}$	-15691.79	-16417.48	-17194.95
AIC	**31246.76**	31421.58	32796.96	34417.91
BIC	**31366.48**	31441.30	32677.25	34506.13

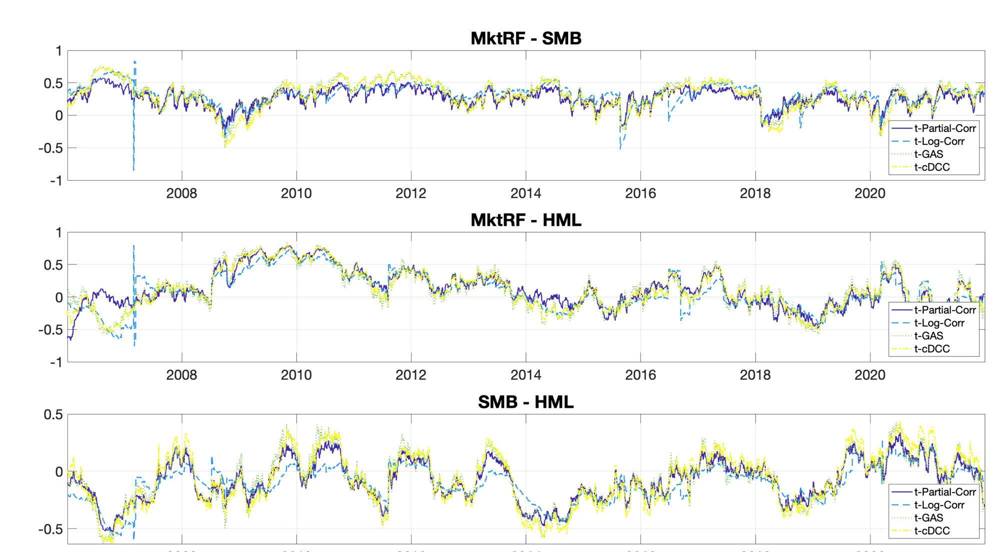

Fig. 3: Filtered conditional correlation coefficients.

Also in the paper...

- An extensive Monte Carlo simulation and an out-of-sample analysis we find that the new approach outperforms a range of recent alternatives.
- We provide conditions for stationarity, ergodicity and invertibility of our model.